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On a hidden symmetry of a relativistic Coulomb problem in the quasipotential approach

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Abstract. The Runge-Lenz vector for a relativistic Coulomb problem in the quasipotential approach is found. The energy spectrum is obtained by an algebraic method. A hidden symmetry group of this problem is shown to be a group O(4) for $|E|<1$, SO(3, 1) for $|E|>1$ and a motion group of the three-dimensional Euclidean space for $|E|=1$.

On the basis of the geometrical properties of the equations for quasipotential approach (Logunov and Tavkhelidze 1963, Kadyshevsky 1968) in the momentum space, the notion of three-dimensional relativistic configurational $\boldsymbol{r}$ representation was introduced by Kadyshevsky *et al* (1968). It was then used to construct a relativistic dynamical scheme in the two-body problem (Kadyshevsky *et al* 1968, 1969a, b, Freeman *et al* 1969). This scheme possesses many important features of quantum mechanics, but unlike quantum mechanics, a quasipotential equation for the wavefunction of the relative motion is written in a finite-difference form. In the case of a local quasipotential $V(\boldsymbol{r}; E)$ the equation for the wavefunction of spinless particles with equal masses has the form (Freeman *et al* 1969)

$$[H_0+V(\boldsymbol{r}; E)]\Psi(\boldsymbol{r})=E\Psi(\boldsymbol{r}) \tag{1}$$

where the finite-difference operator H_0 is a relativistic free Hamiltonian ($\hbar=m=c=1$)

$$H_0=\cosh(\mathrm{i}\nabla_r)+\frac{\mathrm{i}}{r}\sinh(\mathrm{i}\nabla_r)+\frac{\boldsymbol{L}^2}{2r^2}\exp(\mathrm{i}\nabla_r) \qquad \nabla_r=\frac{\partial}{\partial r} \tag{2}$$

and $\boldsymbol{L}^2$ is the square of the angular momentum operator.

In the relativistic configurational $\boldsymbol{r}$-representation some central quasipotentials, which are the relativistic generalisations of the exactly solvable problems of non-relativistic quantum mechanics, were considered. In particular, Donkov *et al* (1971), Atakishiyev *et al* (1982, 1985) and Atakishiyev (1984) construct and investigate a relativistic model of the three-dimensional harmonic oscillator possessing, as in the non-relativistic case, a high hidden U(3) symmetry and a dynamical SU(1, 1) symmetry group. A motion of the relativistic particle in the attractive Coulomb field is studied by Freeman *et al* (1969). It is described by the equation

$$H\Psi(\boldsymbol{r})\equiv(H_0-\alpha/r)\Psi(\boldsymbol{r})=E\Psi(\boldsymbol{r}) \tag{3}$$

where α is the fine structure constant. Due to the trivial O(3) symmetry here the angular dependences of the wavefunction are separated in the standard manner:

$$\Psi(\boldsymbol{r})=(1/r)\Psi_l(r)Y_{lm}(\vartheta, \varphi). \tag{4}$$

As shown by Freeman *et al* (1969), the discrete energy spectrum of the relativistic Coulomb problem under consideration is defined by the formula†

$$E_n = (1-\alpha^2/n^2)^{1/2} \qquad n = 1, 2, 3, \ldots . \tag{5}$$

The corresponding radial wavefunctions are expressed through the hypergeometric function

$$\Psi_{nl}(r) = C_{nl}(r) \exp(-r\chi_n)(-r)^{(l+1)} F[-n+l+1, -ir+l+1; 2l+2; 1-\exp(-2i\chi_n)] \tag{6}$$

where $C_{ne}(r)$ is an arbitrary i periodic function, $\sin\chi_n = \alpha/n$ and $r^{(\lambda)}$ is the 'generalised degree' introduced by Kadyshevsky *et al* (1969b)

$$r^{(\lambda)} = i^{\lambda} \frac{\Gamma(-ir+\lambda)}{\Gamma(-ir)}.$$

However, in the case of the continuous spectrum, when $E = \cosh\chi$

$$\Psi_{El}(r) = C_{El}(r)\, e^{ir\chi}(-r)^{(l+1)} F\left(l+1-\frac{i\alpha}{\sinh\chi}, -ir+l+1; 2l+2; 1-e^{-2\chi}\right). \tag{7}$$

It is to be emphasised that the expression (5) possesses the correct non-relativistic limit $E_n - 1 \to -\alpha^2/2n^2$ and is independent of the orbital quantum number l taking the values $0, 1, 2, \ldots, n-1$. Thus, in complete analogy with the non-relativistic case, an 'accidental' degeneracy of the energy levels with respect to the orbital quantum number takes place.

It is well known that the 'accidental' degeneracy of the non-relativistic hydrogen atom was first explained by Fock as far back as 1935. He has found that the Hamiltonian of the hydrogen atom is invariant under the four-dimensional rotation group O(4) (for $E<0$) or under the group of transformations isomorphic to the Lorentz group (for $E>0$) (Fock 1935a,b) (see also Bander and Itzykson 1966, Perelomov and Popov 1966). As was shown by Bargmann (1935) the hidden symmetry of the hydrogen atom is closely connected with the existence of the additional, besides the angular momentum $\boldsymbol{L}$, integral of motion—the Runge–Lenz vector (Pauli 1926)

$$\boldsymbol{A}_N = \sqrt{m}\left(\alpha\frac{\boldsymbol{r}}{r} + \frac{1}{2m}(\boldsymbol{L}\times\boldsymbol{p}_N - \boldsymbol{p}_N\times\boldsymbol{L})\right) \tag{8}$$

where $\boldsymbol{p}_N$ is the non-relativistic momentum operator. Just the operators $\boldsymbol{L}$ and $\boldsymbol{A}_N$ generate the hidden symmetry group of the Hamiltonian of the hydrogen atom and the energy levels of the bound states are completely defined by their algebra.

The purpose of the present paper is to find a hidden symmetry group (an invariance group) of the relativistic Coulomb problem under consideration. In the cases when $|E|<1$, $E\neq 0$ (the discrete spectrum) and $E=0$ it was group O(4); and in the case when $|E|>1$ (the continuous spectrum) the Lorentz group SO(3, 1) turned out to be such a group. In the special case with $|E|=1$ a hidden symmetry group is isomorphic to the motion group of the three-dimensional Euclidean space.

It will be noted that the relativistic Coulomb problems having the O(4) symmetry are also considered by Biedenharn and Swamy (1964), Itzykson *et al* (1970) and Barut and Bornzin (1973).

† As is shown here the energy spectrum is actually symmetric relative to the $E=0$ point.

We can show that the 'accidental' degeneracy of the relativistic Coulomb problem considered here, similar to the problem on the non-relativistic hydrogen atom, is connected with the existence, besides the operator $\boldsymbol{L}$, of one more integral of motion

$$\boldsymbol{A}=\alpha\left[\left(\frac{i}{r}-1\right)\boldsymbol{n}-\frac{1}{r}\boldsymbol{m}\right]+\tfrac{1}{2}(\boldsymbol{p}\times\boldsymbol{L}-\boldsymbol{L}\times\boldsymbol{p}) \tag{9}$$

where $\boldsymbol{p}$ is a finite-difference momentum operator in the relativistic configurational $\boldsymbol{r}$ representation. (The expressions for the momentum $\boldsymbol{p}$ as well as for the three-dimensional vectors $\boldsymbol{n}$ and $\boldsymbol{m}$ are presented in the appendix.) In the non-relativistic limit the integral of motion $\boldsymbol{A}$ coincides with the Runge-Lenz vector $\boldsymbol{A}_N$ (8) and thus is a generalisation of the vector $\boldsymbol{A}_N$ for the relativistic case.

Using the formulae given in the appendix it can be verified by direct calculation that the commutation relations between the operators $\boldsymbol{L}$, $\boldsymbol{A}$ and H assume the following form:

$$\begin{aligned}&[L_i, L_j]=i\varepsilon_{ijk}L_k \qquad [L_i, A_j]=i\varepsilon_{ijk}A_k\\&[A_i, A_j]=i(1-H^2)\varepsilon_{ijk}L_k\\&[L_i, H]=[A_i, H]=0 \qquad i, j, k=1,2,3.\end{aligned} \tag{10}$$

As in the non-relativistic case it is also true that

$$\boldsymbol{AL}=\boldsymbol{LA}=0. \tag{11}$$

The square of the vector $\boldsymbol{A}$ is expressed through the squares of the angular momentum operator $\boldsymbol{L}^2$ and the Hamiltonian H^2 as

$$A^2=(H^2-1)(\boldsymbol{L}^2+1)+\alpha^2. \tag{12}$$

Going now to the operators

$$N_i=\begin{cases}A_i\,(1-H^2)^{1/2} & \text{for } |E|<1\\ A_i & \text{for } |E|=1\\ A_i(H^2-1)^{-1/2} & \text{for } |E|>1\end{cases} \tag{13}$$

instead of (10) we obtain

$$\begin{aligned}&[L_i, L_j]=i\varepsilon_{ijk}L_k \qquad [L_i, N_j]=i\varepsilon_{ijk}N_k\\&[N_i, N_j]=\sigma i\varepsilon_{ijk}L_k \qquad [N_i, H]=0\end{aligned} \tag{14}$$

where $\sigma=1, 0, -1$ for the ranges $|E|<1$, $|E|=1$ and $|E|>1$, respectively. Equations (11) and (12) are written in terms of the operators N_i in the form

$$\begin{aligned}&\boldsymbol{LN}=\boldsymbol{NL}=0\\&(1-H^2)(\boldsymbol{L}^2+\sigma\boldsymbol{N}^2+1)=\alpha^2 \qquad \sigma=\pm1\\&\boldsymbol{N}^2=(H^2-1)(\boldsymbol{L}^2+1)+\alpha^2 \qquad \sigma=0.\end{aligned} \tag{15}$$

According to the values of $\sigma=\pm1, 0$ the commutation relations correspond to the three different Lie algebras.

(i) For $\sigma=1$ ($|E|<1$) the relations (14) coincide with the commutation relations for the generators of the rotation group O(4) of the four-dimensional Euclidean space. In this case to define the discrete energy levels of the relativistic Coulomb problem (3) by an algebraic method, as usual we introduce two angular momentum operators $\boldsymbol{J}^{(1)}$ and $\boldsymbol{J}^{(2)}$, which commute among themselves as

$$\begin{aligned} &\boldsymbol{J}^{(1)}=\tfrac{1}{2}(\boldsymbol{L}+\boldsymbol{N}) \qquad &&\boldsymbol{J}^{(2)}=\tfrac{1}{2}(\boldsymbol{L}-\boldsymbol{N}) \\ &[J_i^{(a)}, J_j^{(a)}]=i\varepsilon_{ijk}J_k^{(a)} \qquad &&a=1,2. \end{aligned} \tag{16}$$

In accordance with the relations (15) we have

$$\begin{aligned} &(\boldsymbol{J}^{(1)})^2=(\boldsymbol{J}^{(2)})^2=\tfrac{1}{4}(\boldsymbol{L}^2+\boldsymbol{N}^2)=j(j+1) \\ &|E_j|=\left(1-\frac{\alpha^2}{(2j+1)^2}\right)^{1/2}. \end{aligned} \tag{17}$$

Comparing (5) with (17), we conclude that $n=2j+1$, i.e. $j=\frac{1}{2}(n-1)=0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. Thus, the states of the discrete spectrum of the relativistic Coulomb problem, corresponding to the fixed value of the principal quantum number n, are described with the same finite-dimensional irreducible representation $D(\frac{1}{2}(n-1), \frac{1}{2}(n-1))$ of the group O(4) as the states of the discrete spectrum of the hydrogen atom.

(ii) For $\sigma=-1$ ($|E|>1$) the relations (14) take the form of the commutation relations of the Lie algebra of the Lorentz group SO(3, 1). As is known, the values of the invariant Casimir operators $F=\boldsymbol{L}^2-\boldsymbol{N}^2$ and $G=\boldsymbol{L}\boldsymbol{N}$ for the infinite-dimensional irreducible unitary representation of the principal series $D(m, \rho)$ are equal to

$$F=-[1+\tfrac{1}{4}(\rho^2-m^2)] \qquad G=\tfrac{1}{4}m\rho$$

where m is an integer number and $0<\rho<\infty$. In the case of the continuous spectrum of the relativistic Coulomb problem from (15) we obtain that

$$F=-\left(1+\frac{\alpha^2}{E^2-1}\right) \qquad G=0$$

i.e. $m=0$ and $\rho=2\alpha/(E^2-1)^{1/2}$. Consequently, the wavefunctions of the continuous spectrum with a given energy E realise a representation $D(0, \rho)$ of the Lorentz group.

(iii) For $\sigma=0$ ($|E|=1$) the components of the relativistic Runge-Lenz operator, as follows from (14), commute among themselves. In this case an algebra of the operators $\boldsymbol{L}$ and $\boldsymbol{A}$ coincides with the algebra of motion group of the three-dimensional Euclidean space.

Thus, we have found the hidden symmetry group of the relativistic Coulomb problem in the quasipotential approach and defined its energy spectrum by the group-theoretic method.

In conclusion we note that an explicit form of the wavefunctions with $E=\pm 1, 0$ can be obtained from (17) by taking the corresponding limit $E\to\pm 1, 0$. For instance, for $E=\pm 1$ we have

$$\begin{aligned} &\Psi_l^{\pm}(r)=\varphi^{\pm}(r)(-r)^{(l+1)}\phi(-ir+l+1, 2l+2; \mp 2i\alpha) \\ &\varphi^{+}(r)=c_1(r)/r \qquad \varphi^{-}(r)=c_2(r)\,\mathrm{e}^{-\pi r}/r \end{aligned} \tag{18}$$

where $\phi(a, b; x)$ is the confluent hypergeometric function.

We also note that each of the energy eigenvalues $E=\pm 1$, 0, as in the continuous spectrum case, is infinitely degenerated.

Appendix

Here we present some formulae used in the text. A momentum operator in the relativistic configurational $\boldsymbol{r}$ representation, which was found by Kadyshevsky *et al* (1969a), can be written in spherical coordinates in the following compact form:

$$\boldsymbol{P} = -\boldsymbol{n}[\exp(\mathrm{i}\nabla_r) - H_0] - \boldsymbol{m}(1/r)\exp(\mathrm{i}\nabla_r) \tag{A1}$$

where $\boldsymbol{n} = \boldsymbol{r}/r$ is a unit vector along the radius vector $\boldsymbol{r}$:

$$\boldsymbol{n} = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta) \tag{A2}$$

and a three-dimensional vector $\boldsymbol{m}$ has the following components:

$$\begin{aligned} m_1 &= i\left(\cos\varphi\cos\vartheta\frac{\partial}{\partial\vartheta} - \frac{\sin\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right) \\ m_2 &= i\left(\sin\varphi\cos\vartheta\frac{\partial}{\partial\vartheta} + \frac{\cos\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right) \\ m_3 &= -i\sin\vartheta\frac{\partial}{\partial\vartheta}. \end{aligned} \tag{A3}$$

In the limiting case when the velocity of light $c\to\infty$, $\boldsymbol{p}$ goes into the non-relativistic momentum operator

$$\boldsymbol{p} \to \boldsymbol{p}_N = -\left(i\boldsymbol{n}\nabla_r + \boldsymbol{m}\frac{1}{r}\right). \tag{A4}$$

It can be shown that the scalar and vector products of the independent vector operators $\boldsymbol{L}$, $\boldsymbol{n}$ and $\boldsymbol{m}$ are equal to

$$\begin{aligned} (a)\qquad & \boldsymbol{m}^2 = \boldsymbol{L}^2 \qquad \boldsymbol{nL} = \boldsymbol{Ln} = 0 \\ & \boldsymbol{mL} = \boldsymbol{Lm} = 0 \qquad \boldsymbol{nm} = 0 \qquad \boldsymbol{mn} = 2i \end{aligned} \tag{A5}$$

$$\begin{aligned} (b)\qquad & \boldsymbol{n}\times\boldsymbol{L} = \boldsymbol{m} \qquad \boldsymbol{L}\times\boldsymbol{n} = 2i\boldsymbol{n} - \boldsymbol{m} \\ & \boldsymbol{m}\times\boldsymbol{L} = i\boldsymbol{m} - \boldsymbol{n}\boldsymbol{L}^2 \qquad \boldsymbol{L}\times\boldsymbol{m} = i\boldsymbol{m} + \boldsymbol{n}\boldsymbol{L}^2 \\ & \boldsymbol{n}\times\boldsymbol{m} = -\boldsymbol{L} \qquad \boldsymbol{m}\times\boldsymbol{n} = \boldsymbol{L}. \end{aligned} \tag{A6}$$

For the components of these operators the following commutation relations hold $(i, j, k = 1, 2, 3)$:

$$\begin{aligned} [n_i, L_j] &= i\varepsilon_{ijk}n_k \qquad [m_i, L_j] = i\varepsilon_{ijk}m_k \\ [m_i, m_j] &= -i\varepsilon_{ijk}L_k \qquad [n_i, m_j] = i(n_i n_j - \delta_{ij}). \end{aligned} \tag{A7}$$

At the same time we have

$$[\boldsymbol{L}^2, n_i] = 2(n_i + im_i) \qquad [\boldsymbol{L}^2, m_i] = -2in_i\boldsymbol{L}^2. \tag{A8}$$

It is easy also to verify by direct calculation that

$$[p_i, p_j] = [p_i, H_0] = 0. \tag{A9}$$

Under Hermitian conjugation with respect to the scalar product $\int \Psi_1^*\Psi_2\,\mathrm{d}\boldsymbol{r}$ we have $n_i^+ = n_i$, $m_i^+ = m_i - 2in_i$.

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